

Light-front potential for heavy quarkonia constrained by the holographic soft-wall model

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We derive a light-front Schrödinger-type equation of motion for the quark-antiquark wave function of heavy quarkonia imposing constraints from the holographic soft-wall model.

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I. INTRODUCTION

The main objective of this paper is to derive a Schrödinger type equation of motion (EOM) for the quark-antiquark wave function of heavy quarkonia imposing constraints from the holographic soft-wall model. Recently this topic received additional interest because of the discovery of gauge-gravity duality, which helps to establish an equivalence between gravity on anti-de Sitter (AdS) space and light-front QCD – light-front holography [1]. Using light-front holography one can derive light-front wave functions for light and heavy hadrons [2]–[9]. The light-front wave functions obey a Schrödinger-type EOM with an effective confinement potential which is quadratically dependent on the transverse coordinate. This EOM produces linear Regge-trajectories for hadronic masses. Recently, in Ref. [10], it was shown that this quadratic potential is related to an instant-form (IF) effective potential (after squaring). It is consistent with effective potentials obtained in lattice gauge theory and from string models of hadrons, including an approximate agreement with the numerical value of the strength of these potentials of $\simeq 400$ MeV. Another interesting application of the gauge/gravity duality to heavy-quark potentials has been pursued in Ref. [11].

Here we would like to continue our analysis of the light-front (LF) Schrödinger-type equation for heavy quarkonia started in Refs. [5]–[7]. In particular, in these papers we derived a LF Schrödinger-type equation for mesons, containing light and heavy quarks, from a holographic soft-wall model [5]–[7],[12]–[14]. The corresponding wave functions produce the correct Regge-behavior of hadronic mass spectra and are consistent with constraints imposed

by chiral symmetry in the light quark sector and heavy quark symmetry in the heavy quark sector. In particular, for light pseudoscalar mesons the masses satisfy the Gell-Mann-Oakes-Renner and Gell-Mann-Okubo relations. In the sector of heavy quarks we get agreement with heavy quark effective theory and potential models for heavy quarkonia. In the heavy quark mass limit $m_Q \rightarrow \infty$ we obtain the correct scaling of leptonic decay constants both for heavy-light mesons $f_{Q\bar{q}} \sim 1/\sqrt{m_Q}$ and for heavy quarkonia $f_{Q\bar{Q}} \sim \sqrt{m_Q}$ and $f_{c\bar{b}} \sim m_c/\sqrt{m_b}$ with $m_c \ll m_b$. In this limit we also generate the correct expansion of heavy meson masses

$$\begin{aligned} M_{Q\bar{q}} &= m_Q + \bar{\Lambda} + \mathcal{O}(1/m_Q), \\ M_{Q_1\bar{Q}_2} &= m_{Q_1} + m_{Q_2} + E + \mathcal{O}(1/m_{Q_{1,2}}), \end{aligned} \quad (1)$$

where the quantities $\bar{\Lambda}$ and E are the $\mathcal{O}(1)$ contributions to the masses of heavy mesons. Their splittings, e.g. between the vector and pseudoscalar states of heavy-light mesons, become

$$M_{Q\bar{q}}^V - M_{Q\bar{q}}^P \sim \frac{1}{m_Q}. \quad (2)$$

Notice that in Refs. [5]–[7] the effective LF potential consists of two parts: a nonperturbative part – with a confining potential dictated by the holographic soft-wall model, and a perturbative part – the color Coulomb potential. It is important to note that the presence of these two parts in the LF potential has been confirmed by a recent analysis performed in Ref. [10].

In this paper, using our results obtained in Refs. [5]–[7] and those of Ref. [10], we focus on the connection between the IF potential (the sum of a linearly rising confinement

potential and the perturbative color Coulomb potential) and the potential in LF QCD, used for the evaluation of LF wave functions and for predictions of hadronic mass spectra. For simplicity we concentrate on the case of heavy quarkonia systems.

II. LIGHT-FRONT EQUATION OF MOTION FOR HEAVY QUARKONIA

We start from the IF EOM in three-dimensional momentum space, describing a bound state of a heavy quark Q and an antiquark \bar{Q}

$$2E_k \psi(\mathbf{k}) + \int \frac{d^3\mathbf{k}'}{(2\pi)^3 E_{k'}} V(\mathbf{k} - \mathbf{k}') \psi(\mathbf{k}') = M \psi(\mathbf{k}) \quad (3)$$

or in matrix form

$$\begin{aligned} & \int \frac{d^3\mathbf{k}}{(2\pi)^3 E_k} \psi^\dagger(\mathbf{k}) (2E_k - M) \psi(\mathbf{k}) \\ &= - \int \frac{d^3\mathbf{k} d^3\mathbf{k}'}{(2\pi)^6 E_k E_{k'}} \psi^\dagger(\mathbf{k}) V(\mathbf{k} - \mathbf{k}') \psi(\mathbf{k}'). \end{aligned} \quad (4)$$

M and m_Q are the masses of the heavy meson and heavy quark/antiquark, respectively, $\psi(\mathbf{k})$ is the IF wave function and $E_k = \sqrt{\mathbf{k}^2 + m_Q^2}$ is the heavy quark energy. We work in the rest frame of the heavy quarkonia with

$$\begin{aligned} \mathbf{k}_Q &= -\mathbf{k}_{\bar{Q}} = \mathbf{k}, \\ \mathbf{k}_{\perp, Q} &= -\mathbf{k}_{\perp, \bar{Q}} = \mathbf{k}_\perp, \\ k_Q^3 &= -k_{\bar{Q}}^3 = k^3. \end{aligned} \quad (5)$$

The IF wave function obeys the normalization condition

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3 E_k} |\psi(\mathbf{k})|^2 = 1. \quad (6)$$

$V(\mathbf{k} - \mathbf{k}')$ is the effective IF potential in momentum space. Its Fourier transform $V(\mathbf{r})$ in coordinate space has a very clear interpretation (see recent discussion in Ref. [10] in terms of the nonperturbative part of the Cornell potential [15]), which is a linearly rising confinement potential confirmed by QCD lattice calculations as

$$\begin{aligned} \frac{V(\mathbf{k} - \mathbf{k}')}{E_k E_{k'}} &= \int d^3\mathbf{r} e^{i\mathbf{r}(\mathbf{k} - \mathbf{k}')} V(\mathbf{r}), \\ V(\mathbf{r}) &\equiv V_{\text{nonpert}}(r) = V_0 + \sigma r. \end{aligned} \quad (7)$$

V_0 is a constant term and σ is a parameter, which is identified with the string tension in lattice QCD (see e.g. the discussion in Ref. [16]).

This equation can be written in the LF frame using the relation between IF $\psi(\mathbf{k})$ and LF $\psi(x, \mathbf{k}_\perp)$ wave functions (see discussion about this issue e.g. in Refs. [17, 18])

$$\psi(\mathbf{k}) \rightarrow \sqrt{x(1-x)} \psi(x, \mathbf{k}_\perp) \quad (8)$$

and the expression for the three-momentum component k^3 in terms of \mathbf{k}_\perp and the light-cone coordinates [19] (see details in Appendix A.1)

$$k^3 = \frac{x - 1/2}{\sqrt{x(1-x)}} \sqrt{\mathbf{k}_\perp^2 + m_Q^2}. \quad (9)$$

The Jacobian of the variable transformation $k^3 \rightarrow x$ is

$$\frac{\partial k^3}{\partial x} = \frac{E_k}{2x(1-x)}, \quad (10)$$

which leads to a change of the integration measure

$$\frac{d^3\mathbf{k}}{E_k} \rightarrow \frac{d^2\mathbf{k}_\perp dx}{2x(1-x)}. \quad (11)$$

The effective potential in IF transforms into the effective potential in the LC frame as

$$V(\mathbf{k} - \mathbf{k}') \rightarrow V(x, \mathbf{k}_\perp - \mathbf{k}'_\perp) \delta(1 - x - x') x(1-x), \quad (12)$$

where the delta-function $\delta(1 - x - x')$ imposes that $x + x' = 1$ up to $1/m_Q$ corrections (see details in Appendix A.2). Therefore, in the LF frame the EOM for heavy quarkonia reads

$$\begin{aligned} & \int_0^1 dx \int \frac{d^2\mathbf{k}_\perp}{16\pi^3} \psi^\dagger(x, \mathbf{k}_\perp) \left(\sqrt{\frac{\mathbf{k}_\perp^2 + m_Q^2}{x(1-x)}} - M \right) \psi(x, \mathbf{k}_\perp) \\ &= - \int_0^1 dx \int \frac{d^2\mathbf{k}_\perp d^2\mathbf{k}'_\perp}{(16\pi^3)^2} \psi^\dagger(x, \mathbf{k}_\perp) V(x, \mathbf{k}_\perp - \mathbf{k}'_\perp) \psi(x, \mathbf{k}'_\perp). \end{aligned} \quad (13)$$

In conjugated two-dimensional impact space the LF EOM reads

$$\begin{aligned} & \int_0^1 dx \int d^2\mathbf{b}_\perp \psi^\dagger(x, \mathbf{b}_\perp) \left(\sqrt{\frac{-\partial_{\mathbf{b}_\perp}^2 + m_Q^2}{x(1-x)}} - M \right) \psi(x, \mathbf{b}_\perp) \\ &= - \int_0^1 dx \int d^2\mathbf{b}_\perp \psi^\dagger(x, \mathbf{b}_\perp) V(x, \mathbf{b}_\perp) \psi(x, \mathbf{b}_\perp). \end{aligned} \quad (14)$$

The corresponding normalization conditions for the light-front wave functions are given as

$$\begin{aligned} & \int_0^1 dx \int \frac{d^2\mathbf{k}_\perp}{16\pi^3} |\psi(x, \mathbf{k}_\perp)|^2 = 1, \\ & \int_0^1 dx \int d^2\mathbf{b}_\perp |\psi(x, \mathbf{b}_\perp)|^2 = 1. \end{aligned} \quad (15)$$

Next we derive the squared EOMs. For this aim we drop the terms of order \mathbf{k}_\perp^2/m_Q or $\partial_{\mathbf{b}_\perp}^2/m_Q$, which appear in the $1/m_Q$ expansion of the mixed terms

$$\begin{aligned} V(x, \mathbf{k}_\perp - \mathbf{k}'_\perp) \sqrt{\mathbf{k}_\perp^2 + m_Q^2} &= V(x, \mathbf{k}_\perp - \mathbf{k}'_\perp) m_Q + \dots \\ V(x, \mathbf{b}_\perp) \sqrt{-\partial_{\mathbf{b}_\perp}^2 + m_Q^2} &= V(x, \mathbf{b}_\perp) m_Q + \dots \end{aligned} \quad (16)$$

Using the counting of kinematical variables in the $1/m_Q$ expansion, as discussed in Appendix A.2, the squared EOM (up to order $1/m_Q$ corrections) in impact space reads

$$M^2 = \int_0^1 dx \int d^2 \mathbf{b}_\perp \psi^\dagger(x, \mathbf{b}_\perp) \hat{\mathcal{M}}^2 \psi(x, \mathbf{b}_\perp),$$

$$\hat{\mathcal{M}}^2 = -\frac{\partial_{\mathbf{b}_\perp}^2}{x(1-x)} + \left[\frac{m_Q}{\sqrt{x(1-x)}} + V(x, \mathbf{b}_\perp) \right]^2 \quad (17)$$

where $\hat{\mathcal{M}}^2$ is the effective LF operator producing the mass squared of heavy quarkonia. The effective operator $\hat{\mathcal{M}}^2$ contains three terms — the invariant mass operator of the two-partonic state $\hat{\mathcal{M}}_0^2$

$$\hat{\mathcal{M}}_0^2 = \frac{-\partial_{\mathbf{b}_\perp}^2 + m_Q^2}{x(1-x)}, \quad (18)$$

the squared potential

$$U(x, \mathbf{b}_\perp) = V^2(x, \mathbf{b}_\perp) \quad (19)$$

and a mixed term

$$2V(x, \mathbf{b}_\perp) \frac{m_Q}{\sqrt{x(1-x)}}. \quad (20)$$

In Ref. [10] it was stressed that the soft-wall potential corresponds to the squared part of the nonperturbative Cornell potential. This is true when we consider the massless case and drop the constant term V_0 in the Cornell potential.

In the present manuscript we derive an effective heavy quarkonia potential for finite values of the heavy quark masses using constraints imposed by soft-wall AdS/QCD and the Cornell potential. We remind that the meson mass spectrum was studied in detail in the soft-wall AdS/QCD model in [1, 2, 6, 7]. It was proposed to introduce the holographic coordinate ζ which is related to the impact variable \mathbf{b}_\perp via a holographic mapping [1, 2]

$$\zeta^2 = \mathbf{b}_\perp^2 x(1-x). \quad (21)$$

Then the effective light-front wave function $\psi(x, \zeta)$, depending on the ζ and x variables is factorized into transverse $\Phi(\zeta)$ and longitudinal $f(x, m_Q)$ modes [1, 2, 6, 7]

$$\psi(x, \mathbf{b}_\perp) \rightarrow \psi(x, \zeta), \quad (22)$$

$$\psi(x, \zeta) \sqrt{\frac{2\pi\zeta}{x(1-x)}} \equiv \Phi(x, \zeta) = \Phi(\zeta) f(x, m_Q),$$

where

$$\Phi(\zeta) = \kappa^{1+L} \sqrt{\frac{2n!}{(n+L)!}} \zeta^{1/2+L} e^{-\kappa^2 \zeta^2/2} L_n^L(\kappa^2 \zeta^2), \quad (23)$$

and κ is the dimensional dilaton parameter. In the massless case the longitudinal mode fulfills $f(x, 0) = 1$ [1, 2].

We studied the massive case $m_Q \neq 0$ in Ref. [7] and showed that the choice of the longitudinal wave function $f(x, m_Q)$ can absorb the contribution of the mixing term $2V(\zeta)m_Q/\sqrt{x(1-x)}$ in the effective potential. At the same time this procedure guarantees the agreement of leptonic decay constants and masses of heavy quarkonia with the heavy-quark mass expansion (see discussion in Sec.I). In particular, the required form for the longitudinal mode $f(x, m_Q)$ is

$$f(x, m_Q) = N x^\alpha (1-x)^\alpha, \quad \alpha = \frac{m_Q}{4E} - 1 \quad (24)$$

where E is the parameter defining the $\mathcal{O}(m_Q^0)$ contribution to the heavy quarkonia mass and N is a normalization constant fixed by the condition

$$\int_0^1 dx f^2(x, m_Q) = 1. \quad (25)$$

The equation for the wave function $\Phi(x, \zeta)$ then reads

$$M^2 = \int_0^1 dx \int_0^\infty d\zeta \Phi^\dagger(x, \zeta) \hat{\mathcal{M}}^2 \Phi(x, \zeta), \quad (26)$$

$$\hat{\mathcal{M}}^2 = -\partial_\zeta^2 + \frac{m_Q^2}{x(1-x)} + V^2(\zeta).$$

The expression for the corresponding operator $\hat{\mathcal{M}}^2$ in soft-wall AdS/QCD model [1]-[6] has the form

$$\hat{\mathcal{M}}^2 = -\partial_\zeta^2 + \frac{m_Q^2}{x(1-x)} + \frac{4L^2 - 1}{4\zeta^2} + 2\kappa^2(J-1) + U_{\text{SW}}(\zeta). \quad (27)$$

Comparing the two mass operators $\hat{\mathcal{M}}^2$, in the general case of Eq. (17) and in the soft-wall model (27), following conclusions can be reached: 1) both expressions have the same kinetic term; 2) the terms $(4L^2 - 1)/4\zeta^2$ and $2\kappa^2(J-1)$ in the soft-wall approach clearly take into account the L and J dependence of the hadronic mass operator; 3) as was shown in Ref. [10], the soft-wall AdS/QCD potential $U_{\text{SW}}(\zeta)$ corresponds to the square of the linearized part of the Cornell potential, $U_{\text{SW}}(\zeta)$ has the form $U_{\text{SW}}(\zeta) = \kappa^4 \zeta^2$ [1, 2, 5, 6]; 4) apart from the nonperturbative terms there are important corrections generated by the perturbative part of the Cornell potential — the color Coulomb potential $V_{\text{pert}} = -\frac{4}{3}\alpha_s/r$. We included such corrections in our previous papers [6, 7] in the form of a constant term [20] with

$$M_{\text{pert}}^2 = -\frac{64}{9} \frac{\alpha_s^2 m_Q^2}{(n+L+1)^2}. \quad (28)$$

These corrections are quite important in order to describe for example deviations of the mass trajectories of bottom quarkonia from the Regge-like ones.

Now we are in a position to write down the heavy quarkonia mass operator squared including constraints imposed both by the soft-wall AdS/QCD model and the Cornell potential

$$\begin{aligned}\hat{\mathcal{M}}^2 = & -\partial_\zeta^2 + \frac{m_Q^2}{x(1-x)} + \frac{4L^2-1}{4\zeta^2} + 2\kappa^2(J-1) \\ & + \kappa^4\zeta^2 - \frac{64}{9} \frac{\alpha_s^2 m_Q^2}{(n+L+1)^2}.\end{aligned}\quad (29)$$

The extension to heavy quarks with different flavors $Q_1 \neq Q_2$ is straightforward. One should use the longitudinal wave function

$$\begin{aligned}f(x, m_{Q_1}, m_{Q_2}) &= Nx^{\alpha_1}(1-x)^{\alpha_2}, \\ \alpha_{Q_i} &= \frac{m_{Q_i}}{4E} \left(1 - \frac{E}{2(m_{Q_1} + m_{Q_2})}\right)\end{aligned}\quad (30)$$

which reduces to $f(x, m_Q)$ for $m_{Q_1} = m_{Q_2}$. The expression for the mass operator squared is modified as

$$\begin{aligned}\hat{\mathcal{M}}^2 = & -\partial_\zeta^2 + \frac{m_{Q_1}^2}{x} + \frac{m_{Q_2}^2}{1-x} + \frac{4L^2-1}{4\zeta^2} + 2\kappa^2(J-1) \\ & + \kappa^4\zeta^2 - \frac{64}{9} \frac{\alpha_s^2 m_{Q_1} m_{Q_2}}{(n+L+1)^2}.\end{aligned}\quad (31)$$

Finally, the mass spectrum for heavy quarkonia is given by

$$\begin{aligned}M_{Q_1\bar{Q}_2}^2 = & 4\kappa^2 \left(n + \frac{L+J}{2}\right) \\ & + (1+2\alpha_1+2\alpha_2) \left(\frac{m_{Q_1}^2}{2\alpha_1} + \frac{m_{Q_2}^2}{2\alpha_2}\right) \\ & - \frac{64}{9} \frac{\alpha_s^2 m_{Q_1} m_{Q_2}}{(n+L+1)^2}.\end{aligned}\quad (32)$$

This master formula can be further simplified when we drop the $O(1/m_Q)$ corrections resulting in

$$\begin{aligned}M_{Q_1\bar{Q}_2}^2 = & 4\kappa^2 \left(n + \frac{L+J}{2}\right) + (m_{Q_1} + m_{Q_2} + E)^2 \\ & - \frac{64}{9} \frac{\alpha_s^2 m_{Q_1} m_{Q_2}}{(n+L+1)^2} + \mathcal{O}(1/m_{Q_{1,2}}).\end{aligned}\quad (33)$$

Our results for the mass spectrum of $c\bar{c}$, $b\bar{b}$ and $c\bar{b}$ quarkonia are shown in Table I. We present our predictions for ground and excited states for different values of n , L and J . In the numerical analysis we use the following set of QCD and model parameters: charm and bottom quark masses

$$m_c = 1.275 \text{ GeV}, \quad m_b = 4.18 \text{ GeV}, \quad (34)$$

strong coupling constants

$$\alpha_s(c\bar{c}) = 0.45, \quad \alpha_s(c\bar{b}) = 0.383, \quad \alpha_s(b\bar{b}) = 0.27, \quad (35)$$

TABLE I: Masses of heavy quarkonia $c\bar{c}$, $b\bar{b}$ and $c\bar{b}$

Meson	J^P	n	L	S	Mass [MeV]			
$\eta_c(2980)$	0^-	0,1,2,3	0	0	2975	3477	3729	3938
$\psi(3097)$	1^-	0,1,2,3	0	1	3097	3583	3828	4032
$\chi_{c0}(3415)$	0^+	0,1,2,3	1	1	3369	3628	3843	4038
$\chi_{c1}(3510)$	1^+	0,1,2,3	1	1	3477	3729	3938	4129
$\chi_{c2}(3555)$	2^+	0,1,2,3	1	1	3583	3828	4032	4219
$\eta_b(9390)$	0^-	0,1,2,3	0	0	9337	9931	10224	10471
$\Upsilon(9460)$	1^-	0,1,2,3	0	1	9460	10048	10338	10581
$\chi_{b0}(9860)$	0^+	0,1,2,3	1	1	9813	10110	10359	10591
$\chi_{b1}(9893)$	1^+	0,1,2,3	1	1	9931	10224	10471	10700
$\chi_{b2}(9912)$	2^+	0,1,2,3	1	1	10048	10338	10581	10808
$B_c(6277)$	0^-	0,1,2,3	0	0	6277	6719	6892	7025

parameters defining the $\mathcal{O}(m_Q^0)$ contributions to heavy quarkonia masses

$$\begin{aligned}E_{cc} &= 0.795 \text{ GeV}, \quad E_{cb} = 1.25 \text{ GeV}, \\ E_{bb} &= 1.45 \text{ GeV},\end{aligned}\quad (36)$$

and the dilaton parameters

$$\begin{aligned}\kappa_{cc} &= 0.610 \text{ GeV}, \quad \kappa_{cb} = 0.629 \text{ GeV}, \\ \kappa_{bb} &= 1.079 \text{ GeV}.\end{aligned}\quad (37)$$

Note the masses of the ground states of mesons in Table I are for the fit of free parameters, while the results for the excited states are our predictions.

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Appendix A: Details on kinematical variables: relations and $1/m_Q$ expansions

1. Relation of k^3 and x

First we comment on the derivation of Eq. (9). It follows from a formula relating x and k^3 in the constituent quark rest frame [19]

$$x = \frac{1}{2} + \frac{k^3}{2E_k}. \quad (A1)$$

Then, when substituting

$$E_k = \sqrt{m_Q^2 + \mathbf{k}^2} = \sqrt{m_Q^2 + \mathbf{k}_\perp^2 + (k^3)^2} \quad (\text{A2})$$

we get

$$k^3 = (2x - 1)E_k = (2x - 1)\sqrt{m_Q^2 + \mathbf{k}_\perp^2 + (k^3)^2} \quad (\text{A3})$$

and

$$k^3 = \frac{x - 1/2}{\sqrt{x(1-x)}} \sqrt{\mathbf{k}_\perp^2 + m_Q^2}. \quad (\text{A4})$$

2. Counting of momentum variables in the heavy quark mass expansion

Here we present the counting of momentum variables in the heavy quark mass expansion

$$\begin{aligned} \mathbf{k}_\perp &\sim \mathcal{O}(1), \quad k^3 \sim \mathcal{O}(1), \quad E_k = m_Q + \mathcal{O}(1/m_Q) \\ x &= \frac{1}{2} + \frac{k^3}{2E_k} = \frac{1}{2} + \mathcal{O}(1/m_Q), \quad x - \frac{1}{2} = \mathcal{O}(1/m_Q) \\ \frac{1}{2} + \frac{k^3}{2E_k} &= \frac{1}{2} + \mathcal{O}(1/m_Q), \quad \sqrt{x(1-x)} = \frac{1}{4} + \mathcal{O}(1/m_Q^2), \\ x + x' &= 1 + \frac{k^3}{2E_k} + \frac{k'^3}{2E_{k'}} = 1 + \mathcal{O}(1/m_Q). \end{aligned} \quad (\text{A5})$$

Using the last formula in Eq.(A5) we introduce the delta-function $\delta(1 - x - x')$ which imposes $1 = x + x'$ up to $1/m_Q$ corrections.

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